MATH 312
Lecture 9: Quartet

DIMENSIONS AND THE FUNDAMENTAL THEOREM

<u>Last time:</u>

- Given vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$, consider

$$A = [\vec{v}_1, \dots, \vec{v}_n], \text{ and:}$$

1. $\vec{v}_1, \dots, \vec{v}_m$ are "$\boxed{\text{linearly independent}}$" $\iff$ RREF(A) has a pivot in every <u>column</u>.

Remember This?

2. $\vec{v}_1, \dots, \vec{v}_m$ "$\boxed{\text{span}}$" V $\iff$ RREF(A) has a pivot in every <u>row</u>.

<u>Def</u> $\vec{v}_1, \dots, \vec{v}_n$ form a "BASIS" of V if

a) they are linearly independent, and

b) they span V

<u>So</u> $\vec{v}_1, \dots, \vec{v}_m$ is a <u>basis</u> of V

by a) by b)

$\Updownarrow$

RREF(A) has a pivot in every column <u>AND</u> every row

$\Updownarrow$

$$\text{RREF}(A) = \text{identity matrix, id} = \underbrace{\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}}_{m} \Big\} m$$

$\Updownarrow$

A is <u>INVERTIBLE</u>!

Examples

"HARDER"

a) obviously, $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ is a basis for $\mathbb{R}^3$, called the STANDARD BASIS. Similarly, one can define the standard basis in $\mathbb{R}^n$.

b) Any permutation & scaling of the standard basis, eg $\begin{bmatrix}0\\0\\5\end{bmatrix} \begin{bmatrix}-\frac{1}{2}\\0\\0\end{bmatrix} \begin{bmatrix}0\\8\\0\end{bmatrix}$.

c) Any upper-triangular $n \times n$ matrix with no zeros on the diagonal has columns which form a basis: eg $\begin{bmatrix}4&3&2\\0&4&0\\0&0&6\end{bmatrix}$.

d) Similarly lower triangular...

e) $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$ is a basis of $\mathbb{R}^2$, since $\begin{bmatrix}1&2\\1&1\end{bmatrix}$ is invertible.

f) Infinitely many examples...

IMPORTANT

Any basis of $\mathbb{R}^n$ must contain EXACTLY n vectors! If we have more, then they can not be independent. If we have less, then they can not span!

Def

The "DIMENSION" of a vector space is the number of vectors in any basis of that space.

Def

The "Rank" of a matrix is $\boxed{\dim C(A)}$ = # Pivots in RREF

dimension of col. spc

The Point: A basis is the ultimate in "data-compression": given a FINITE set of vectors, we can produce any other vector as a linear combination UNIQUELY.

eg: $\begin{pmatrix}4\\3\\7\end{pmatrix} = 4\begin{pmatrix}1\\0\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\0\end{pmatrix} + 7\begin{pmatrix}0\\0\\1\end{pmatrix}$ ← This is always possible since a basis spans!

And if $\begin{pmatrix}4\\3\\7\end{pmatrix} = a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix}$,

then $(4-a)\begin{pmatrix}1\\0\\0\end{pmatrix} + (3-b)\begin{pmatrix}0\\1\\0\end{pmatrix} + (7-c)\begin{pmatrix}0\\0\\1\end{pmatrix} = \vec{0}$,

So, $4-a = 3-b = 7-c = 0$ ← This always works by linear independence!

FTLA:

FUNDAMENTAL THEOREM (OF LINEAR ALGEBRA)

First, remember TRANSPOSES: (rows ⟷ columns)

$$A = \begin{bmatrix}2 & 3 & 5\\1 & 9 & 0\end{bmatrix}, \quad A^T = \begin{bmatrix}2 & 1\\3 & 9\\5 & 0\end{bmatrix}$$

and of course, $(A^T)^T = A$.

Now, $A: \mathbb{R}^3 \to \mathbb{R}^2$ and $A^T: \mathbb{R}^2 \to \mathbb{R}^3$

And now, we have $\underline{4}$ subspaces to keep track of!

1. $N(A)$ = Nullspace of A.
2. $C(A)$ = Column Space of A

Already know these!

3. $N(A^T)$ = "left Nullspace" of A
4. $C(A^T)$ = "Row Space" of A

kind of know these too...

SANITY CHECK! If $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 9 & 0 \end{bmatrix} : \mathbb{R}^3 \to \mathbb{R}^2$

Then $N(A)$ and $C(A^T)$ are subspaces of $\mathbb{R}^2$

And $N(A^T)$ and $C(A)$ are subspaces of $\mathbb{R}^3$.

More generally if A is $m \times n$ (m rows and n columns), then:

$$\begin{tikzcd}
N(A) \arrow[r, "\text{is a subspace}"'] & \mathbb{R}^n & C(A^T) \arrow[l, "\text{is a subspace}"] \\
C(A) \arrow[r, "\text{is a subspace}"] & \mathbb{R}^m & N(A^T) \arrow[l, "\text{is a subspace}"']
\end{tikzcd}$$

Here's the FTLA:

Given an $m \times n$ matrix $A : \mathbb{R}^n \to \mathbb{R}^m$, if its RANK (i.e. $\dim C(A)$) is r, then:

- $\dim N(A) = n - r$
- $\dim C(A^T) = r$
- $\dim N(A^T) = m - r$

Again, it all comes down to <u>RREF</u>!!

eg, if $m=3$ and $n=4$, and <u>RREF</u> is

$$\begin{array}{cc} & \text{col} \to \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} \text{Row} \downarrow \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

<u>Note</u>: the transpose of this is <u>NOT</u> in RREF

Then:

known already Add up to "4"

- Columns 1 and 2 (with pivots) <u>don't</u> form a basis for $C(A)$, but count its dimension.
- Columns 3 and 4 (with no pivots) count <u>free variables</u> which produce $N(A)$

New. Add up to "3"

- Rows 1 and 2 (with pivots) form a basis for $C(A^T)$.
- Row 3 counts a free variable which counts the dimension of $N(A^T)$

<u>MORAL</u>

The Number of independent (i.e., pivot) columns equals the number of independent (i.e., pivot) rows! So, if the rows of some $n \times n$ matrix are independent, then so are its columns (& vice-versa).

THIS IS <u>NOT</u> "OBVIOUS".